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# On an integral equation of viscous flow theory

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SUMMARY

The integral equation encountered by van de Vooren and Veldman [1] in their study of the Knudsen region near the leading edge of a flat plate is solved by the method of Wiener and Hopf. This exact solution yields the values of certain arbitrary constants which were not determined in [1].

## 1. Introduction

In a recent investigation of the incompressible viscous flow near the leading edge of a flat plate van de Vooren and Veldman [1] found that, when the Knudsen number $\kappa$ is small, the solution in the immediate neighbourhood of the leading edge depends on that of the integral equation

$$f(x) = (2\pi)^{-1} \int_0^\infty \log|x - x_1| f(x_1) dx_1 + x^{-\frac{1}{2}}. \tag{1.1}$$

The function $f(x)$ is related to the slip velocity on the plate. As a complement to their solutions of the complete Navier–Stokes equations for various values of $\kappa$ the authors presented a numerical solution of (1.1). In addition they found the analytic form of the asymptotic expansions of $f(x)$ to be

$$f(x) = x^{-\frac{1}{2}} + c_1 + \frac{c_1}{2\pi} x \log x + c_2 x + \frac{c_1}{16\pi^2} x^2 (\log x)^2$$
$$+ \frac{1}{4\pi}\left(c_2 - \frac{c_1}{4\pi}\right) x^2 \log x + c_3 x^2 + \ldots, \text{ as } x \to 0, \tag{1.2}$$

and

$$f(x) = x^{-\frac{1}{2}} \left\{ -\frac{1}{\pi} \frac{\log x}{x} + \frac{C_1}{x} - \frac{3}{2\pi^2}\left(\frac{\log x}{x}\right)^2 + \frac{1}{\pi}\left(3C_1 + \frac{2}{\pi}\right)\frac{\log x}{x^2}\right.$$
$$\left. + \frac{C_2}{x^2} + \ldots \right\}, \text{ as } x \to \infty. \tag{1.3}$$

Here $c_i$, $C_i$ are constants which they were unable to determine except by comparison with the numerical solution which gives $-1.265$ as an estimate for $c_1$. The importance of $c_1$

lies in the fact that it determines the slip velocity $u$ at the leading edge which is given by

$$\frac{u}{U} = -\frac{A\kappa^{\frac{1}{2}}}{2} \lim_{x\to 0} \{x^{-\frac{1}{2}} - f(x)\}. \tag{1.4}$$

Here $U$ is the mainstream velocity and $A$ (= 0.755) is a known constant determined by the solution valid away from the immediate neighbourhood of the leading edge.

In the following we present an exact solution of (1.1) obtained by the method of Wiener and Hopf. The asymptotic expansions for small and large $x$ of this solution confirm the form of (1.2), (1.3) and in addition yield the values of the unknown constants. We find those which appear explicitly in (1.2), (1.3); in particular the value of $c_1$ is

$$c_1 = -\left(\frac{\pi}{2}\right)^{\frac{1}{2}}. \tag{1.5}$$

The estimate $c_1 = -1.265$, obtained in [1] agrees well with this exact value ($c_1 = -1.2533$).

## 2. The integral equation

In order to be able to employ complex variable methods we extend (1.1) to negative values of $x$ and $x_1$ by writing it as

$$f(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \log|x - x_1| f(x_1) dx_1 + m(x) + h(x) \tag{2.1}$$

where

$$f(x) = 0, \quad m(x) = 0, \qquad x < 0, \tag{2.2}$$

and

$$h(x) = 0, \quad m(x) = x^{-\frac{1}{2}}, \quad x > 0. \tag{2.3}$$

If the Fourier transform of $f(x)$, for example, is denoted by $F(\omega)$ so that

$$F(\omega) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{i\omega x} f(x) dx, \tag{2.4}$$

a Fourier transform of (2.1) yields

$$F(\omega) = V(\omega)F(\omega) + M(\omega) + H(\omega). \tag{2.5}$$

Here $F(\omega)$, $M(\omega)$ are regular in the upper half-plane im $\omega > 0$, and $H(\omega)$ is regular in the lower half-plane im $\omega < 0$. The function $(2\pi)^{\frac{1}{2}} V(\omega)$ is the transform of $\log x$ which is given by, on introduction of a suitable convergence factor,

$$\begin{aligned} 2\pi\, V(\omega) &= \int_0^{\infty} (\log x) e^{i\omega x - \varepsilon x} dx + \int_{-\infty}^{0} \log|x|\, e^{i\omega x + \varepsilon x} dx \\ &= i\,\frac{\log(\omega - i\varepsilon) + \frac{1}{2}i\pi + \gamma}{\omega - i\varepsilon} - i\,\frac{\log(\omega + i\varepsilon) - \frac{1}{2}i\pi + \gamma}{\omega + i\varepsilon}. \end{aligned} \tag{2.6}$$

Here $\gamma$ is Euler's constant, and the parameter $\varepsilon$ will be allowed to tend to zero in conclusion. The corresponding expression for $M(\omega)$ is

$$(2\pi)^{\frac{1}{2}} M(\omega) = \int_0^\infty x^{-\frac{1}{2}} e^{i\omega x - \varepsilon x} dx = \pi^{\frac{1}{2}} e^{i\pi/4} (\omega + i\varepsilon)^{-\frac{1}{2}}. \tag{2.7}$$

The branches of the square root in (2.7) and of the logarithms in (2.6) are selected so that

$$\log(\omega \pm i\varepsilon) = \log|\omega \pm i\varepsilon| + i \arg(\omega \pm i\varepsilon), \tag{2.8}$$

where we define

$$-\frac{3\pi}{2} \leqq \arg(\omega - i\varepsilon) < \frac{\pi}{2}, \quad -\frac{\pi}{2} \leqq \arg(\omega + i\varepsilon) < \frac{3\pi}{2}, \tag{2.9}$$

and consider the $\omega$-plane to be cut from $i\varepsilon$ to $i\infty$ and from $-i\varepsilon$ to $-i\infty$ along the imaginary axis. Equation (2.5) is now written as

$$F(\omega)\{1 - V(\omega)\} = M(\omega) + H(\omega) \tag{2.10}$$

and this constitutes a Wiener–Hopf problem for the unknown functions $F(\omega)$, $H(\omega)$. To determine these functions we first make the assumption, which may be justified *a posteriori*, that the domain of regularity of $H(\omega)$ extends to $\operatorname{im} \omega < \varepsilon$. Then (2.10) may be written

$$F_+(\omega)\{1 - V(\omega)\} = M_+(\omega) + H_-(\omega) \tag{2.11}$$

where a plus sign indicates that a function is regular for $\operatorname{im} \omega > 0$ and a minus sign for $\operatorname{im} \omega < \varepsilon$. It emerges that, except for $F_+(\omega)$, all functions with the subscript $+$ are in fact regular for $\operatorname{im} \omega > -\varepsilon$.

The procedure is now as follows. It will be shown in the following section that $1 - V(\omega)$ has zeroes at $\omega = \pm\omega_0$ where $\omega_0$ is real. Because of this we define

$$Q(\omega) = \{1 - V(\omega)\}/(\omega^2 - \omega_0^2), \tag{2.12}$$

and factorise $Q(\omega)$ into the form

$$Q(\omega) = Q_+(\omega)/Q_-(\omega), \tag{2.13}$$

where $Q_+(\omega)$ is regular and non-zero for $\operatorname{im} \omega > -\varepsilon$ and $Q_-(\omega)$ is regular and non-zero for $\operatorname{im} \omega < \varepsilon$. We then decompose the product $Q_-(\omega)M_+(\omega)$ so that

$$Q_-(\omega)M_+(\omega)/(\omega - i\varepsilon) = L_+(\omega) - L_-(\omega). \tag{2.14}$$

The reason for the introduction of the factor $\omega - i\varepsilon$ will be evident later. Equation (2.11) may then be arranged as

$$(\omega^2 - \omega_0^2)F_+(\omega)Q_+(\omega) - (\omega - i\varepsilon)L_+(\omega) = H_-(\omega)Q_-(\omega) - (\omega - i\varepsilon)L_-(\omega), \tag{2.14}$$

where the left-hand side represents a function regular for $\operatorname{im} \omega > 0$, and the right-hand side represents a function regular for $\operatorname{im} \omega < \varepsilon$. Since they are equal on a dense set of points, by analytic continuation together they define a function $E(\omega)$ which is regular in the whole $\omega$-plane. In Sections 3, 4 we find that $Q_+(\omega) \sim \omega^{-1}$, $L_+(\omega) \sim e^{i\pi/4}/(2\omega)^{\frac{1}{2}}$ for large $|\omega|$. Now as noted in [1] the first term in the asymptotic expansion of $f(x)$ as $x \to 0$

must be $x^{-\frac{1}{2}}$; thus $E(\omega)$ is a constant, $E_0$ say, so that $F_+(\omega) \sim e^{i\pi/4}/(2\omega)^{\frac{1}{2}}$ for large $|\omega|$. Also in Sections 3, 4 we find that $Q_+(\omega) \sim 2^{-\frac{1}{2}} e^{i\pi/4}\, \omega^{-\frac{3}{2}}$, $\pi L_+(\omega) \sim -\log\omega$ as $\omega \to 0$, so that $F_+(\omega) \sim E_0 2^{\frac{1}{2}} e^{-i\pi/4}\, \omega^{-\frac{1}{2}}$ as $\omega \to 0$. Unless $E_0 = 0$ this implies that $f(x) = O(x^{-\frac{1}{2}})$ as $x \to \infty$ which is incorrect, either from [1] or as may be deduced on examination of (1.1). It therefore follows that each side of (2.15) is zero and hence that

$$F_+(\omega) = \frac{(\omega - i\varepsilon)L_+(\omega)}{(\omega^2 - \omega_0^2)Q_+(\omega)}. \tag{2.16}$$

The solution may then be completed by inverting the Fourier transforms and taking the limit as $\varepsilon \to 0$.

## 3. The decomposition of the function $Q(\omega)$

The function $Q$ defined in (2.12) is a one-valued function of the complex variable $\omega$ in the plane cut in accord with the definitions (2.9). It is easy to show that $1 - V(\omega)$ has no zeroes when $|\omega| = O(1)$ but it emerges that it has two zeroes on the real axis with $|\omega| = O(\varepsilon \log \varepsilon)$ as $\varepsilon \to 0$. This may be seen as follows. When $\omega$ is real and positive we have from (2.6) that

$$2\pi\{1 - V(\omega)\} = 2\pi + \{2\varepsilon\gamma + \pi\omega + \varepsilon \log(\omega^2 + \varepsilon^2) - 2\omega \tan^{-1}(\varepsilon/\omega)\}/(\omega^2 + \varepsilon^2). \tag{3.1}$$

When $|\omega| \gg 1$ the right-hand side is positive since $2\pi$ is the dominant term. However when $|\omega| \ll \varepsilon$ the right-hand side is dominated by a term $(\log \varepsilon^2)/\varepsilon$ which is negative. Thus the expression has a zero which is found to be at $\omega = \omega_0$ where

$$\omega_0 \sim \frac{2\varepsilon}{\pi} \log\left(\frac{\pi}{2\varepsilon}\right) \quad \text{as } \varepsilon \to 0. \tag{3.2}$$

Similarly it may be shown that $1 - V(\omega)$ also has a zero at $-\omega_0$.

The function $Q(\omega)$ is then regular and also non-zero in the strip $-\varepsilon < \operatorname{im} \omega < \varepsilon$. To effect its decomposition we consider $Q'(\omega)/Q(\omega)$ which is regular in the strip and also tends to zero as $\operatorname{re} \omega \to \pm\infty$ with $-\varepsilon < \operatorname{im} \omega < \varepsilon$. We may therefore write

$$Q'(\omega)/Q(\omega) = Q'_+(\omega)/Q_+(\omega) - Q'_-(\omega)/Q_-(\omega) = R_+(\omega) - R_-(\omega), \tag{3.3}$$

where

$$R_+(\omega) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{Q'(z)}{Q(z)} \frac{dz}{z - \omega}, \quad R_-(\omega) = \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{Q'(z)}{Q(z)} \frac{dz}{z - \omega} \tag{3.4}$$

for $-\varepsilon < c < \operatorname{im} \omega < d < \varepsilon$.

For the evaluation of $R_+(\omega)$ we first perform a partial integration yielding

$$R_+(\omega) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{\log Q(z)}{(z - \omega)^2}\, dz, \tag{3.5}$$

and then replace the path of integration indicated in (3.4) by the infinite semi-circle below the real axis, the two sides of the cut from $-i\varepsilon$ to $-i\infty$ and a small circle surrounding the point $z = -i\varepsilon$. Only the cut gives a contribution to $R_+(\omega)$. Since the result will be required only in the limit $\varepsilon \to 0$ we present it after making the limiting process. The retention of $\varepsilon$ so that $R_+(\omega)$ is written as a function of $\omega + i\varepsilon$ serves merely as a reminder that $R_+(\omega)$ is regular for im $\omega > -\varepsilon$. The result is

$$R_+(\omega) = -\frac{3}{2(\omega + i\varepsilon)} + \frac{2i}{\pi\{1 - 4(\omega + i\varepsilon)^2\}}$$
$$\times \{\log(\omega + i\varepsilon) + \log 2 - \tfrac{1}{2}i\pi + i\pi(\omega + i\varepsilon)\}. \tag{3.6}$$

Similarly we evaluate $R_-(\omega)$ by replacing the path of integration by a contour above the real axis and obtain finally

$$R_-(\omega) = \frac{3}{2(\omega - i\varepsilon)} + \frac{2i}{\pi\{1 - 4(\omega - i\varepsilon)^2\}}$$
$$\times \{\log(\omega - i\varepsilon) + \log 2 + \tfrac{1}{2}i\pi - i\pi(\omega - i\varepsilon)\}. \tag{3.7}$$

The functions $Q_+(\omega)$, $Q_-(\omega)$ may now be obtained from (3.6), (3.7) by integration and we choose the arbitrary multiplicative constant so that

$$Q_+(\omega) \sim (\omega + i\varepsilon)^{-1}, \quad Q_-(\omega) \sim \omega - i\varepsilon \tag{3.8}$$

for $|\omega|$ large. We shall find in particular that we require the values of $Q_-(-it)$, $Q_+(it)$ where $t$ is real and positive. These are quite easily found to be

$$Q_-(-it) = -\frac{i2^{\frac{1}{2}}t^{\frac{3}{2}}}{(1 + 4t^2)^{\frac{1}{4}}} \exp\left[\frac{2}{\pi}\int_0^t \frac{\log 2\lambda}{1 + 4\lambda^2} d\lambda\right] = -\frac{1}{Q_+(it)}. \tag{3.9}$$

The two properties of $Q_+(\omega)$ which were quoted in Section 2 to deduce that the two sides of equation (2.15) were zero are now evident. The first, that $Q_+(\omega) \sim \omega^{-1}$ as $\omega \to \infty$, is given in (3.8), and the second, that $Q_+(\omega) \sim 2^{-\frac{1}{2}} e^{i\pi/4} \omega^{-\frac{3}{2}}$ as $\omega \to 0$, follows either directly on integration of (3.6) or from (3.9) on replacing $it$ by $\omega$ with the appropriate value of $\pi/2$ for arg $\omega$.

## 4. The decomposition of $Q_-(\omega)M_+(\omega)/(\omega - i\varepsilon)$

As stated in Section 2 the function $Q_-(\omega)M_+(\omega)/(\omega - i\varepsilon)$ is to be decomposed in the form (2.14). The factor $\omega - i\varepsilon$ is inserted to ensure that the function tends to zero as re $\omega \to \pm\infty$. Formulae analogous to (3.4) lead to

$$L_+(\omega) = -\frac{1}{\pi i}\int_0^\infty \frac{Q_-(-it)\,dt}{2^{\frac{1}{2}}t^{\frac{3}{2}}\{t - i(\omega + i\varepsilon)\}}, \tag{4.1}$$

$$L_-(\omega) = -\frac{1}{\pi i}\int_0^\infty \frac{2^{\frac{1}{2}}t^{\frac{3}{2}}Q_+(it)\,dt}{(1 + 4t^2)\{t + i(\omega - i\varepsilon)\}}, \tag{4.2}$$

where $Q_-(-it)$, $Q_+(it)$ are given by (3.9). Again the limiting process $\varepsilon \to 0$ has been made except where it serves as a reminder of the domain of analyticity of $L_+(\omega)$, $L_-(\omega)$.

The two properties of $L_+(\omega)$ used following (2.15) in Section 2 are now available. When

$t \gg 1$ we have from (3.9) that $Q_-(-it) \sim -it$, and it then follows from (4.1) that

$$L_+(\omega) \sim \frac{1}{\pi(2\omega)^{\frac{1}{2}}} \int_0^\infty \frac{dp}{p^{\frac{1}{2}}(p-i)} = \frac{e^{i\pi/4}}{(2\omega)^{\frac{1}{2}}} \quad \text{as } \omega \to \infty. \tag{4.3}$$

However when $t \ll 1$ we see that $Q_-(-it) \sim -i2^{\frac{1}{2}}t^{\frac{3}{2}}$ with the result that

$$L_+(\omega) \sim (-\log \omega)/\pi \quad \text{as } \omega \to 0. \tag{4.4}$$

## 5. Calculation of $f(x)$

We may now return to (2.16) and invert the Fourier transform to obtain $f(x)$. The functions $L_+(\omega)$, $Q_+(\omega)$ are regular, and $Q_+(\omega)$ is non-zero, for im $\omega > -\varepsilon$. However the presence of the factor $\omega^2 - \omega_0^2$ in the numerator of (2.16) means that $F_+(\omega)$ is regular only for im $\omega > 0$. The inversion formula therefore gives

$$(2\pi)^{\frac{1}{2}} f(x) = \int_{-\infty}^{\infty} \frac{(\omega - i\varepsilon)L_+(\omega)}{(\omega^2 - \omega_0^2)Q_+(\omega)} e^{-i\omega x} d\omega \tag{5.1}$$

where the path of integration must be taken *above* the poles of the integrand on the real axis at $\omega = \pm\omega_0$. However as shown in Section 3, 4 we have that $Q_+(\omega) = O(\omega^{-\frac{3}{2}})$, $L_+(\omega) = O(\log \omega)$ as $\omega \to 0$, from which it follows that the residues at these poles are both $O(\omega_0^{\frac{3}{2}} \log \omega_0)$, and therefore tend to zero in the limit $\varepsilon \to 0$ since $\omega_0 = O(\varepsilon \log \varepsilon)$.

We replace the contour of integration in (5.1) by the infinite semi-circle below the real axis together with the two sides of the cut from $-i\varepsilon$ to $-i\infty$ along the imaginary axis. To simplify the form of the integrand on the sides of the cut we replace it by, on use of (2.12), (2.14),

$$\left\{ \frac{(\omega - i\varepsilon)L_-(\omega)}{Q_-(\omega)} + M_+(\omega) \right\} \frac{e^{-i\omega x}}{1 - V(\omega)} \tag{5.2}$$

which enables us to exploit the analyticity of $L_-(\omega)$, $Q_-(\omega)$ in the lower half-plane. As explained above, we may ignore the residues at the two poles at $\omega = \pm\omega_0$ within the contour. The result, on using (3.9), (4.2) and finally letting $\varepsilon \to 0$, is

$$(2\pi)^{\frac{1}{2}} f(x) = \int_0^\infty \left[ \frac{s}{1+s^2} - \frac{e^{m(s)}}{\pi(1+s^2)^{\frac{3}{4}}} \int_0^\infty \frac{e^{m(t)}}{(1+t^2)^{\frac{3}{4}}} \frac{dt}{t+s} \right] s^{\frac{1}{2}} e^{-sx/2} ds \tag{5.3}$$

where

$$m(t) = -\frac{1}{\pi} \int_0^t \frac{\log \lambda}{1+\lambda^2} d\lambda. \tag{5.4}$$

It is shown in the Appendix that

$$\int_0^\infty \frac{e^{m(t)}}{(1+t^2)^{\frac{3}{4}}} \frac{dt}{t+s} = \frac{\pi}{s} \left\{ 1 - \frac{e^{-m(s)}}{(1+s^2)^{\frac{1}{4}}} \right\}, \tag{5.5}$$

so that (5.3) may be written

$$f(x) = \frac{1}{x^{\frac{1}{2}}} - \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \frac{e^{m(s)}}{s^{\frac{1}{2}}(1+s^2)^{\frac{3}{4}}} e^{-sx/2} ds. \tag{5.6}$$

It is fairly easy to verify, by the methods outlined in the Appendix for the evaluation of the integral in (5.5), that $f(x)$ does indeed satisfy (1.1).

## 6. The expansion of $f(x)$ for large $x$

The asymptotic expansion of $f(x)$ for large $x$ follows easily from (5.6). To obtain this we need the expansion of the integrand for small $s$ which is, on use of (5.4),

$$\frac{e^{m(s)}}{s^{\frac{1}{2}}(1+s^2)^{\frac{3}{4}}} = s^{-\frac{1}{2}}\left\{1 + \frac{s}{\pi}(1-\log s) + \frac{\frac{1}{2}s^2}{\pi^2}(1-\log s)^2 - \tfrac{3}{4}s^2 + O((s\log s)^3)\right\}. \tag{6.1}$$

Term by term evaluation of (5.6) then leads to the expansion (1.3) as predicted in [1] with

$$C_1 = (1 - \log 2 - \gamma)/\pi, \tag{6.2a}$$

$$C_2 = \frac{3}{2\pi^2}\{(\log 2)^2 - 2(\gamma + \tfrac{1}{3})\log 2 - \gamma^2 + \tfrac{10}{3}\gamma + \tfrac{11}{3} + \pi^2\}. \tag{6.2b}$$

## 7. The expansion of $f(x)$ for small $x$

The value of the constant $c_1$ in (1.2) follows immediately from (5.6). It is

$$c_1 = \lim_{x\to 0}\{f(x) - x^{-\frac{1}{2}}\} = -\frac{1}{(2\pi)^{\frac{1}{2}}}\int_0^\infty \frac{e^{m(s)}}{(1+s^2)^{\frac{3}{4}}} ds \tag{7.1}$$

since, from (5.4),

$$m(s) = m(s^{-1}). \tag{7.2}$$

It is shown in the Appendix that

$$\int_0^\infty \frac{e^{m(s)}}{(1+s^2)^{\frac{3}{4}}} ds = \pi \tag{7.3}$$

where the integrals required in the calculation of $c_2$, $c_3$ are also evaluated. These are

$$\mathscr{F}\int_0^\infty \frac{e^{m(s)}}{(1+s^2)^{\frac{3}{4}}}\frac{ds}{s} = 1, \quad \mathscr{F}\int_0^\infty \frac{e^{m(s)}}{(1+s^2)^{\frac{3}{4}}}\frac{ds}{s^2} = \frac{1}{2\pi} - \frac{5\pi}{12}, \tag{7.4}$$

where $\mathscr{F}$ denotes the finite part of the infinite integral.

The asymptotic expansion of (5.6) as $x \to 0$ is found to be as predicted in [1]. It is (1.2) with

$$c_1 = -\left(\frac{\pi}{2}\right)^{\frac{1}{2}}, \quad c_2 = 2^{-\frac{3}{2}}\pi^{-\frac{1}{2}}(\log 2 - \gamma + 2), \tag{7.5a}$$

$$c_3 = -\frac{2^{-\frac{1}{2}}\pi^{-\frac{3}{2}}}{16}\{(\log 2)^2 + (5 - 2\gamma)\log 2 + \gamma^2 - 5\gamma + \tfrac{15}{2} - \tfrac{2}{3}\pi^2\}. \tag{7.5b}$$

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**Appendix**

We now evaluate the integrals of (5.5), (7.3), (7.4). If we extend the definition of $m(t)$ in (5.4) to complex values of a variable $z$ say, and define $0 \leqq \arg z < 2\pi$, we find that near $z = i$, $e^{-m(z)} \sim 2^{-\frac{1}{4}}(z-i)^{\frac{1}{4}}$ so that $e^{-m(z)}/(1+z^2)^{\frac{1}{4}}$ is regular at $z = i$. Also when $t$ is real and positive

$$e^{-m(-t)} = \frac{1-it}{(1+t^2)^{\frac{1}{2}}} e^{m(t)}. \tag{A.1}$$

To evaluate the integral of (5.5) with $0 < s < \infty$ we consider the function

$$\frac{e^{-m(z)}}{z(1+z^2)^{\frac{1}{4}}(z-s)} \tag{A.2}$$

which is regular for im $z > 0$ in the plane cut along the positive real axis, and has simple poles at $z = 0, s$. The integral of this function around the contour consisting of the infinite semi-circle in the upper half-plane and the real axis indented at $z = 0, s$ is zero. Thus

$$\lim_{\delta \to 0} \left\{ \int_\delta^\infty \frac{e^{m(t)}}{(1+t^2)^{\frac{3}{4}}} \frac{1-it}{t(t+s)} dt + \mathscr{P} \int_\delta^\infty \frac{e^{-m(t)}}{(1+t^2)^{\frac{1}{4}}} \frac{dt}{t(t-s)} + \frac{\pi i}{s} - \frac{\pi i e^{-m(s)}}{s(1+s^2)^{\frac{1}{4}}} \right\} = 0 \tag{A.3}$$

where $\mathscr{P}$ denotes the Cauchy principal value of the integral at $t = s$, and $\delta$ is the radius of the semi-circle about the origin. The imaginary part of (A.3) gives the result (5.5).

To evaluate the integral of (7.3) we consider the function

$$\frac{e^{-m(z)}}{z(1+z^2)^{\frac{1}{4}}} \tag{A.4}$$

around the same contour apart from the indentation at $z = s$. The imaginary part of the expression analogous to (A.3) leads to the required result. For the integrals of (7.4) we consider

$$\frac{e^{-m(z)}}{z^2(1+z^2)^{\frac{1}{4}}}, \quad \frac{e^{-m(z)}}{z^3(1+z^2)^{\frac{1}{4}}} \tag{A.5}$$

around the contour and select the imaginary parts of the finite parts of the integrals around the semi-circle radius $\delta$ about the origin. Both results follow immediately.

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